Limits

A. Simplify the following limits:

1. $\lim_{x \to -2} -3x^5 + 4x^3 - 10$

$= -3(-2)^5 + 4(-2)^3 - 10$

$= 96 - 32 - 10$

$= 54$

2. $\lim_{x \to -3} \frac{6x^2 - 5}{3 - x}$

$= \frac{6(-3)^2 - 5}{3-(-3)}$

$= \frac{54 - 5}{6}$

$= \frac{49}{6}$

3. $\lim_{x \to 0} \frac{x^3 + 64}{x + 4}$

$\lim_{x \to 0} \frac{(x+4)(x^2 - 4x + 16)}{(x+4)}$

$= x^2 - 4x + 16$

$= (-4)^2 - 4(-4) + 16$

$= 48$

4. $\lim_{x \to 6} \frac{x^2 + 2x - 48}{x - 6}$

$\lim_{x \to 6} \frac{(x+8)(x-6)}{(x-6)}$

$= x + 8$

$= 6 + 8$

$= 14$

5. $\lim_{x \to 0} \frac{\sqrt{x+25} + 5}{x}$

$\lim_{x \to 0} \frac{(\sqrt{x+25} + 5)}{x} \cdot \frac{(\sqrt{x+25} - 5)}{(\sqrt{x+25} - 5)}$

$\lim_{x \to 0} \frac{x + 25 - 25}{x(\sqrt{x+25} - 5)}$

$\lim_{x \to 0} \frac{1}{\sqrt{x+25} - 5}$

$= \frac{1}{\sqrt{0+25} - 5}$

$= \frac{1}{5-5} = \frac{1}{0} =$ *undefined*

6. $\lim_{x \to 0} \frac{\sin 6x}{5x}$

$\lim_{x \to 0} \frac{\frac{6}{5} \cdot \sin 6x}{\frac{6}{5} \cdot 5x}$

$\frac{6}{5} \lim_{x \to 0} \frac{\sin 6x}{6x}$

$= \frac{6}{5} \cdot 1 = \frac{6}{5}$

7.

$$\lim_{x\to 0} \frac{\sin(2x)\cos^2 x - \sin(2x)\cos^3 x}{(2x)^2}$$

$$\lim_{x\to 0} \frac{\sin(2x)\cos^2 x(1-\cos x)}{(2x)^2}$$

$$\lim_{x\to 0} \frac{\sin(2x)}{(2x)} \cdot \cos^2 x \cdot \frac{(1-\cos x)}{2x}$$

$$= 1 \cdot 1^2 \cdot \frac{1}{2} \cdot 0 = 0$$

8.

$$\lim_{x\to 81} \frac{x-81}{\sqrt{x}-9}$$

$$\lim_{x\to 81} \frac{(\sqrt{x}+9)(\sqrt{x}-9)}{(\sqrt{x}-9)}$$

$$= \sqrt{81} + 9 = 18$$

9.

$$\lim_{x\to\infty} \frac{6x^5 - 2x^4 + 4x^2}{3x^2 + 7x^4 - 11x^5 + x^3}$$

Since the largest exponents are the same in both the numerator an denominator the answer is -6/11

10.

$$\lim_{x\to\infty} \frac{3x^4 - 5x^3 + 3}{10x^4 + 2x^2 - 9}$$

Since the largest exponents are the same in both the numerator an denominator the answer is 3/10

11.

$$\lim_{x\to\infty} \frac{-5x^4 - 5x^3 - 2}{4x^6 + 3x^5 + 2x^4 - 9x}$$

Since the largest exponent is in the denominator the answer is 0

12.

$$\lim_{x\to 6^+} \frac{4}{6-x}$$

Since direct substitution does not work, we note that any substitution of a value a little bigger than 6 results in a very small negative value in the denominator, the result of the limit is negative infinity

13. $$\lim_{x\to -3^+} \frac{1}{x^2 - x - 12}$$

Since direct substitution does not work, we note that any substitution of a value a little bigger than -3 results in a very small negative value in the denominator, the result of the limit is minus infinity

14. $$\lim_{x\to -\infty} -x^4 - 8x^2$$

Direct substitution of any negative number results in a value that is negative so we can assume that substitution of negative infinity will result in an answer of negative infinity

15. $$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

$$\lim_{x\to\infty} \frac{(2x^2+1)^{1/2}}{(3x-5)^{2/2}}$$

$$\lim_{x\to\infty} \left(\frac{2x^2+1}{9x^2-30x+25}\right)^{1/2}$$

$$\left(\lim_{x\to\infty} \left(\frac{2x^2+1}{9x^2-30x+25}\right)\right)^{1/2}$$

$$\left(\frac{2}{9}\right)^{1/2} = \frac{\sqrt{2}}{3}$$

Two possible methods:

1. Convert to exponents, find a common denominator, expand necessary parts

or

2. Divide each part by variable with largest power inside radical and divide each component outside radical with the same variable but with exponent reduced by root index

and

use then use the rules as they apply to infinity

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

$$\lim_{x\to\infty} \frac{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{\frac{3x}{x}-\frac{5}{x}}$$

$$\lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}}$$

$$= \frac{\sqrt{2+0}}{3-0} = \frac{\sqrt{2}}{3}$$

B. Show that $\lim_{x\to\infty} \sqrt{x^2+1} - x = 0$

$$\lim_{x\to\infty} \frac{(\sqrt{x^2+1}-x)}{1} \cdot \frac{(\sqrt{x^2+1}+x)}{(\sqrt{x^2+1}+x)} = \lim_{x\to\infty} \frac{x^2+1-x^2}{(\sqrt{x^2+1}+x)} = \lim_{x\to\infty} \frac{1}{(\sqrt{x^2+1}+x)} = \frac{1}{\infty} = 0$$

A number divided the an infinitely large number yields a result equal to 0

C. Determine the equation of the tangent line to the curve $f(x) = x^3 + 3x - 7$ at x = -1 (do not use derivatives)

a) slope formula

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^3 + 3(x+h) - 7] - [x + 3x - 7]}{h}$$

$$= \lim_{h\to 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3 + 3x + 3h - 7] - [x^3 + 3x - 7]}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 3x + 3h - 7 - x^3 - 3x + 7}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + 3h}{h}$$

$$= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 + 3)}{h} = 3x^2 + 3$$

b) Point of tangency

$$f(x) = (-1)^3 + 3(-1) - 7$$
$$f(x) = -11$$
$$(-1, -11)$$

c) Slope

$$m = 3x^2 + 3$$
$$m = 3(-1)^2 + 3$$
$$m = 6$$

d) Tangent Line

$$(y_2 - y_1) = m(x_2 - x_1)$$
$$(y - (-11)) = 6(x - (-1))$$
$$y + 11 = 6x + 6$$
$$y = 6x - 5$$

D. A ball is thrown into the air with a velocity of 40ft/sec, its height in feet after t seconds is given by $y = 40t - 16t^2$. Find the following:

Slope formula: $$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \to 0} \frac{[40(t+h) - 16(t+h)^2] - [40t - 16t^2]}{h}$$

$$= \lim_{h \to 0} \frac{[40t + 40h - 16t^2 - 32th - h^2] - [40t - 16t^2]}{h}$$

$$= \lim_{h \to 0} \frac{40t + 40h - 16t^2 - 32th - h^2 - 40t + 16t^2}{h}$$

$$= \lim_{h \to 0} \frac{40h - 32th - h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(40 - 32t - h)}{h} = 40 - 32t$$

a) the velocity of the ball at 1 sec.

$$v(t) = 40 - 32t$$
$$v(1) = 40 - 32(1)$$
$$v(1) = 8 \ \ ft/\sec$$

b) velocity at 3 sec

$$v(t) = 40 - 32t$$
$$v(3) = 40 - 32(3)$$
$$v(3) = -56 \ \ ft/\sec$$

c) maximum height reached by the ball

Max height reached when velocity of 0

$$v(t) = 40 - 32t$$
$$0 = 40 - 32t$$
$$-40 = -32t$$
$$5/4 = t$$

Height

$$y = 40t - 16t^2$$
$$y = 40\left(\frac{5}{4}\right) - 16\left(\frac{5}{4}\right)^2$$
$$y = 50 - 25 = 25 \ feet$$

d. the time it takes for the ball to return to the ground

$$y = 40 - 16t^2$$
$$0 = 8t(5 - 2t)$$
$$0 = 8t \text{ or } 0 = 5 - 2t$$
$$t = 0 \text{ or } t = 2.5$$

e) the velocity of the ball when it strikes the ground

$$v(t) = 40 - 32t$$
$$v(2.5) = 40 - 32(2.5)$$
$$v(2.5) = 40 - 80$$
$$v(2.5) = -40 \ ft/\sec$$

E. Identify the three instances when a limit fails to exist

a) when the behavior from the right (+) differs from the behavior from the left (-)

b) when the function increases or decreases without bound - never reaches any real number

c) the function oscillates between 2 fixed values as x approaches c