1.

$$\int \frac{x+2}{x^2-2x+1}dx$$

a) Factor denominator $\Rightarrow (x-1)(x-1) \Rightarrow (x-1)^2$ multiple linear factor

b) Write partial fraction equation $\Rightarrow \frac{A}{(x-1)} + \frac{B}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2} = \frac{Ax-1A+B}{(x-1)^2} = \frac{Ax+\left(-A+B\right)}{(x-1)^2}$

Matching coefficients and constants

$\frac{x+2}{x^2-2x+1} = \frac{Ax+\left(-A+B\right)}{(x-1)^2} \Rightarrow A=1$ and $\left(-A+B\right)=2,$ by substitution $B=3$

$$\therefore \int \frac{1}{(x-1)}dx + \int \frac{3}{(x-1)^2}dx \Rightarrow \int \frac{1}{(x-1)}dx + 3\int \frac{1}{(x-1)^2}dx = \ln(x-1) - 3(x-1)^{-1}$$

$a)\ \int \frac{1}{(x-1)}dx = \int \frac{1}{u}du = \ln u = \ln(x-1)$

$u=(x-1) \Rightarrow du = dx$

$b)\ 3\int \frac{1}{(x-1)^2}dx = 3\int (x-1)^{-2}dx = 3\int u^{-2}du = 3\frac{u^{-1}}{-1} = -3u^{-1} = -3(x-1)^{-1}$

$u=(x-1) \Rightarrow du = dx$

2.

$$\int \frac{1}{x^3+x}dx$$

a) Factor denominator $\Rightarrow x(x^2+1)$ a linear factor and quadratic factor

b) Write partial fraction equation $\Rightarrow \frac{A}{x} + \frac{Bx+C}{(x^2+1)} = \frac{A(x^2+1)+\left(Bx+C\right)x}{x(x^2+1)} = \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}$

Matching coefficients and constants

$\frac{1}{x^3+x} = \frac{(A+B)x^2+Cx+A}{x(x^2+1)} \Rightarrow A+B=0, C=0$ and $A=1,$

by substitution $B=-1 \Rightarrow A+B=0 \Rightarrow 1+B=0 \Rightarrow B=-1 \therefore$

$$\int \frac{1}{x}dx + \int \frac{-1x}{(x^2+1)}dx = \int \frac{1}{x}dx - \int \frac{x}{(x^2+1)}dx = \ln x - \frac{1}{2}\ln(x^2+1)$$

$a)\ \int \frac{1}{x}dx = \ln x$

$b)\ \int \frac{x}{(x^2+1)}dx \Rightarrow \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln u = \frac{1}{2}\ln(x^2+1)$

$u=(x^2+1) \Rightarrow du = 2xdx \Rightarrow \frac{1}{2}du = xdx$

$$\int \frac{x}{(x-4)^2}dx$$

a) Denominator factored $\Rightarrow (x-4)^2$ multiple linear factor

b) Write partial fraction equation $\Rightarrow \dfrac{A}{(x-4)} + \dfrac{B}{(x-4)^2} = \dfrac{A(x-4)+B}{(x-4)^2} = \dfrac{Ax-4A+B}{(x-4)^2}$

Matching coefficients and constants

3. $\dfrac{x}{(x-4)^2} = \dfrac{Ax+\left(-4A+B\right)}{(x-4)^2} \Rightarrow A = 1$ and $-4A + B = 0$,

by substitution $-4A + B = 0 \Rightarrow -4(1) + B = 0 \Rightarrow B = 4 \therefore$

$$\int \frac{1}{(x-4)}dx + \int \frac{4}{(x-4)^2}dx = \int \frac{1}{(x-4)}dx + 4\int \frac{1}{(x-4)^2}dx = \ln(x-4) - 4(x-4)^{-1}$$

a) $\displaystyle\int \frac{1}{(x-4)}dx = \ln(x-4)$

b) $\displaystyle 4\int \frac{1}{(x-4)^2}dx \Rightarrow 4\int (x-4)^{-2}dx \Rightarrow 4\int u^{-2}du \Rightarrow 4\cdot\frac{u^{-1}}{-1} = -4u^{-1} = -4(x-4)^{-1}$

$u = (x-4) \Rightarrow du = dx$

$$\int \frac{x^2+3x-2}{x^3+5x}dx$$

a) Factor denominator $\Rightarrow x(x^2+5)$ a linear factor and quadratic factor

b) Write partial fraction equation $\Rightarrow \dfrac{A}{x} + \dfrac{Bx+C}{(x^2+5)} = \dfrac{A(x^2+5)+\left(Bx+C\right)x}{x(x^2+5)} = \dfrac{Ax^2+5A+Bx^2+Cx}{x(x^2+5)}$

Matching coefficients and constants

$\dfrac{x^2+3x-2}{x^3+5x} = \dfrac{(A+B)x^2+Cx+5A}{x(x^2+5)} \Rightarrow A+B=1, C=3$ and $5A=-2 \Rightarrow A = -\dfrac{2}{5}$

by substitution $B = \dfrac{7}{5} \Rightarrow A+B=1 \Rightarrow -\dfrac{2}{5}+B=1 \Rightarrow -2+5B=5 \Rightarrow 5B=7 \Rightarrow B=\dfrac{7}{5} \therefore$

4. $$\int \frac{\frac{-2}{5}}{x}dx + \int \frac{\frac{7}{5}x+3}{(x^2+5)}dx = \frac{-2}{5}\int \frac{1}{x}dx + \frac{1}{5}\int \frac{7x+15}{(x^2+5)}dx = -\frac{2}{5}\int \frac{1}{x}dx + \frac{1}{5}\int \frac{7x}{(x^2+5)}dx + \frac{1}{5}\int \frac{15}{(x^2+5)}dx =$$

$$-\frac{2}{5}\ln x + \frac{7}{10}\ln(x^2+5) + \frac{3}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}}$$

a) $\displaystyle -\frac{2}{5}\int \frac{1}{x}dx = -\frac{2}{5}\ln x$

b) $\displaystyle \frac{1}{5}\int \frac{7x}{(x^2+5)}dx = \frac{7}{5}\int x(x^2+5)^{-1}dx = \frac{7}{5}\cdot\frac{1}{2}\int u^{-1}du = \frac{7}{10}\ln u = \frac{7}{10}\ln(x^2+5)$

$u = (x^2+5) \Rightarrow du = 2xdx \Rightarrow \dfrac{1}{2}du = xdx$

c) $\displaystyle \frac{1}{5}\int \frac{15}{(x^2+5)}dx = \frac{15}{5}\int \frac{1}{(x^2+5)}dx = 3\int \frac{1}{\left(x^2+(\sqrt{5})^2\right)}dx = 3\cdot\frac{1}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}} = \frac{3}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}}$

5.

$$\int \frac{x^3}{\left(x^2+2\right)^2}dx$$

a) Denominator factored $\Rightarrow (x^2+2)^2$ multiple quadratic factor

b) Write partial fraction equation $\Rightarrow \dfrac{Ax+B}{(x^2+2)}+\dfrac{Cx+D}{(x^2+2)^2}=\dfrac{\left(Ax+B\right)(x^2+2)+Cx+D}{(x^2+2)^2}=$

$$=\frac{Ax^3+2Ax+Bx^2+2B+Cx+D}{(x^2+2)^2}=\frac{Ax^3+Bx^2+(2A+C)x+(2B+D)}{(x^2+2)^2}$$

Matching coefficients and constants

$$\frac{x^3}{\left(x^2+2\right)^2}=\frac{Ax^3+Bx^2+(2A+C)x+(2B+D)}{(x^2+2)^2}\Rightarrow A=1, B=0,\ (2A+C)=0 \text{ and } \left(2B+D\right)=0,$$

by substitution $C=-2\Rightarrow(2A+C)=0\Rightarrow(2(1)+C)=0\Rightarrow 2+C=0\Rightarrow C=-2$

by substitution $D=0\Rightarrow\left(2B+D\right)=0\Rightarrow(2(0)+D)=0\Rightarrow D=0$

$$\int\frac{x}{(x^2+2)}dx+\int\frac{-2x}{(x^2+2)^2}dx=\int\frac{x}{(x^2+2)}dx-2\int\frac{x}{(x^2+2)^2}dx=\frac{1}{2}\ln(x^2+2)+2(x^2+2)^{-1}$$

$a)\ \int\frac{x}{(x^2+2)}dx=\frac{1}{2}\int x\cdot(x^2+2)^{-1}dx=\frac{1}{2}\int u^{-1}du=\frac{1}{2}\ln u=\frac{1}{2}\ln(x^2+2)$

$u=(x^2+2)\Rightarrow du=2xdx\Rightarrow\frac{1}{2}du=xdx$

$b)\ 2\int\frac{x}{(x^2+2)^2}dx=2\int x(x^2+2)^{-2}dx=2\int u^{-2}du=2\frac{u^{-1}}{-1}=-2u^{-1}=-2(x^2+2)^{-1}$

$u=(x^2+2)\Rightarrow du=2xdx\Rightarrow\frac{1}{2}du=xdx$

6.

$$\int \frac{x^2+6x+4}{x^4+5x^2+4}dx$$

a) Factored denominator $\Rightarrow (x^2+1)(x^2+4)$ two different quadratic factors

b) Write partial fraction equation $\Rightarrow \dfrac{Ax+B}{(x^2+1)}+\dfrac{Cx+D}{(x^2+4)}=\dfrac{(Ax+B)(x^2+4)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+4)}=$

$$=\frac{Ax^3+4Ax+Bx^2+4B+Cx^3+Cx+Dx^2+D}{(x^2+1)(x^2+4)}=\frac{(A+C)x^3+(B+D)x^2+(4A+C)x+(4B+D)}{(x^2+1)(x^2+4)}$$

Matching coefficients and constants

$$\frac{x^2+6x+4}{x^4+5x^2+4}=\frac{(A+C)x^3+(B+D)x^2+(4A+C)x+(4B+D)}{(x^2+1)(x^2+4)}\Rightarrow$$

$(A+C)=0,(B+D)=1,(4A+C)=6,(4B+D)=4$

Solve using systems of equations and the process of substitution

1. $(A+C)=0,(4A+C)=6\Rightarrow$

$$A=-C\ \therefore\ 4(-C)+C=6\Rightarrow -3C=6\Rightarrow C=-2 \text{ and } A=2$$

2. $(B+D)=1,(4B+D)=4\Rightarrow$

$$D=1-B\ \therefore\ 4B+(1-B)=4\Rightarrow 3B=3\Rightarrow B=1 \text{ and } D=0$$

$$\int\frac{2x+1}{(x^2+1)}dx+\int\frac{-2x}{(x^2+4)}dx\Rightarrow 2\int\frac{x}{(x^2+1)}dx+\int\frac{1}{(x^2+1)}dx-2\int\frac{x}{(x^2+4)}dx=$$

$\ln(x^2+1)+\tan^{-1}x-\ln(x^2+4)$

a) $2\int\frac{x}{(x^2+1)}dx=2\cdot\frac{1}{2}\int\frac{1}{u}du=\int u^{-1}du=\ln u=\ln(x^2+1)$

$u=(x^2+1)\Rightarrow du=2xdx\Rightarrow\frac{1}{2}du=xdx$

b) $\int\frac{1}{(x^2+1)}dx=\frac{1}{1}\tan^{-1}\frac{x}{1}=\tan^{-1}x$

c) $-2\int\frac{x}{(x^2+4)}dx=-2\cdot\frac{1}{2}\int\frac{1}{u}du=-\int u^{-1}du=-\ln u=-\ln(x^2+4)$

$u=(x^2+4)\Rightarrow du=2xdx\Rightarrow\frac{1}{2}du=xdx$

7.

$$\int \frac{x}{x^4+7x^2+6}dx$$

a) Factored denominator $\Rightarrow (x^2+1)(x^2+6)$ two different quadratic factors

b) Write partial fraction equation $\Rightarrow \dfrac{Ax+B}{(x^2+1)}+\dfrac{Cx+D}{(x^2+6)}=\dfrac{\left(Ax+B\right)(x^2+6)+\left(Cx+D\right)(x^2+1)}{(x^2+1)(x^2+6)}=$

$$=\frac{Ax^3+6Ax+Bx^2+6B+Cx^3+Cx+Dx^2+D}{(x^2+1)(x^2+6)}=\frac{(A+C)x^3+(B+D)x^2+(6A+C)x+(6B+D)}{(x^2+1)(x^2+6)}$$

Matching coefficients and constants

$$\frac{x}{x^4+7x^2+6}=\frac{(A+C)x^3+(B+D)x^2+(6A+C)x+(6B+D)}{(x^2+1)(x^2+6)}$$

$(A+C)=0,(B+D)=0,(6A+C)=1,(6B+D)=0$

Solve using systems of equations and the process of substitution

1. $(A+C)=0,(6A+C)=1\Rightarrow$

$$A=-C\therefore 6(-C)+C=1\Rightarrow -5C=6\Rightarrow C=-\frac{6}{5}\text{ and }A=\frac{6}{5}$$

2. $(B+D)=0,(6B+D)=0\Rightarrow$

$$D=-B\therefore 6B+(-B)=0\Rightarrow 5B=0\Rightarrow B=0\text{ and }D=0$$

$$\int\frac{\frac{6}{5}x}{(x^2+1)}dx+\int\frac{-\frac{6}{5}x}{(x^2+6)}dx=\frac{6}{5}\int\frac{x}{(x^2+1)}dx-\frac{6}{5}\int\frac{x}{(x^2+6)}dx=$$

$$\frac{3}{5}\ln(x^2+1)-\frac{3}{5}\ln(x^2+6)$$

a) $\dfrac{6}{5}\displaystyle\int\frac{x}{(x^2+1)}dx=\frac{6}{5}\int x(x^2+1)^{-1}dx=\frac{6}{5}\cdot\frac{1}{2}\int u^{-1}du=\frac{3}{5}\ln u=\frac{3}{5}\ln(x^2+1)$

$u=(x^2+1)\Rightarrow du=2xdx\Rightarrow \frac{1}{2}du=xdx$

b) $\dfrac{6}{5}\displaystyle\int\frac{x}{(x^2+6)}dx=\frac{6}{5}\int x(x^2+6)^{-1}dx=\frac{6}{5}\cdot\frac{1}{2}\int u^{-1}du=\frac{3}{5}\ln u=\frac{3}{5}\ln(x^2+6)$

$u=(x^2+6)\Rightarrow du=2xdx\Rightarrow \frac{1}{2}du=xdx$

8.

$$\int \frac{x^4 - x^3 - x - 1}{x^3 - x^2} dx$$

Since the degree of the numerator is greater than that of the denominator, the first step is long division, answer $\Rightarrow x - \dfrac{x+1}{x^3 - x^2}$

a) Factor denominator $\Rightarrow x^2(x-1) \Rightarrow x^2$ is multiple linear factor and $(x-1)$ is a single linear factor

b) Write partial fraction equation $\Rightarrow \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{(x-1)} = \dfrac{A(x)(x-1) + B(x-1) + C(x^2)}{x^2(x-1)} =$

$$\frac{Ax^2 - Ax + Bx - B + Cx^2}{x^2(x-1)} = \frac{(A+C)x^2 + (-A+B)x - B}{x^2(x-1)}$$

Matching coefficients and constants

$$\frac{x+1}{x^3 - x^2} = \frac{(A+C)x^2 + (-A+B)x - B}{x^2(x-1)} \Rightarrow (A+C) = 0,\ \left(-A+B\right) = 1 \text{ and } -B = 1 \Rightarrow B = -1$$

by substitution $\left(-A+B\right) = 1 \Rightarrow -A + (-1) = 1 \Rightarrow -A = 2 \Rightarrow A = -2$

by substitution $\left(A+C\right) = 0 \Rightarrow -2 + C = 0 \Rightarrow C = 2$

$$\therefore \int x dx - \left[\int \frac{-2}{x} dx + \int \frac{-1}{x^2} dx + \int \frac{2}{(x-1)} dx\right] = \int x dx + 2\int \frac{1}{x} dx + \int x^{-2} dx - 2\int \frac{1}{(x-1)} dx =$$

$$\frac{x^2}{2} + 2\ln x - x^{-1} - 2\ln(x-1)$$

$a)\ \int x dx = \dfrac{x^2}{2}$

$b)\ 2\int \dfrac{1}{x} dx = 2\int x^{-1} dx = 2\ln x$

$c)\ \int x^{-2} dx = \dfrac{x^{-1}}{-1} = -x^{-1}$

$d)\ -2\int \dfrac{1}{(x-1)} dx = -2\ln(x-1)$