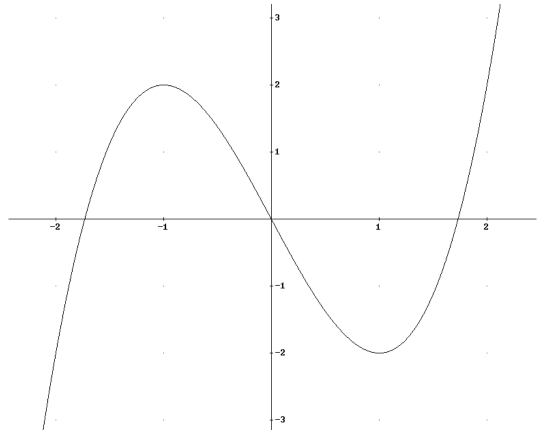
Optimization Problems

1. Find the maximum and minimum values of $f(x) = x^3 - 3x$ on the interval [-2, 2]. {-2, -1,1, 2}

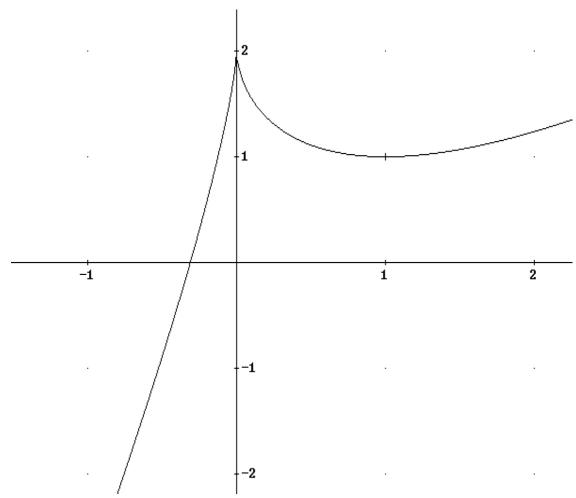
$$f(x) = x^3 - 3x$$
$$f'(x) = 3x^2 - 3 \Rightarrow f'(x) = 3(x^2 - 1) \Rightarrow$$
$$f'(x) = (x-1)(x+1) \Rightarrow x = -1, x = 1$$
$$f(-2) = (-2)^3 - 3(-2) = -8 + 6 = -2$$
$$f(-1) = (-1)^3 - 3(-1) = -1 + 3 = 2$$
$$f(1) = (1)^3 - 3(1) = 1 - 3 = -2$$
$$f(2) = (2)^3 - 3(2) = 8 - 6 = 2$$



2. Find the maximum and minimum values of $f(x) = 2 + 2x - 3x^{\frac{2}{3}}$ on the interval [-1, 2].{-1,0}

$$f(x) = 2 + 2x - 3x^{\frac{2}{3}}$$
$$f'(x) = 2 - 3\left(\tfrac{2}{3}\right)x^{-1/3} \Rightarrow f'(x) = 2 - 2x^{-1/3} \Rightarrow$$
$$f'(x) = 2(1 - x^{-1/3}) \Rightarrow 1 - x^{-1/3} = 0 \Rightarrow -x^{-1/3} = 1 \Rightarrow$$
$$-1/x^{1/3} = 1 \Rightarrow -1 = x^{1/3} \Rightarrow (-1)^3 = \left(x^{1/3}\right)^3 \Rightarrow x = -1$$
$$f(-1) = 2 + 2(-1) - 3(-1)^{\frac{2}{3}} = 2 - 2 - 3 = -3$$
$$f(2) = 2 + 2(2) - 3(2)^{\frac{2}{3}} = 2 + 4 - 3 = 3$$

$$f''(x) = (-2)\left(-1/3\right)x^{-4/3} \Rightarrow 2/3\, x^{-4/3}$$
$$f''(-1) = 2/3\,(-1)^{-4/3} > 0 \text{ concave up}$$
$$f''(1) = 2/3\,(1)^{-4/3} > 0 \text{ concave up}$$
$$f(0) = 2 + 2(0) - 3(0)^{\frac{2}{3}} = 2$$



3. Find two nonnegative numbers whose sum is 10 and the sum of whose squares is a minimum. {5}

Numbers $\Rightarrow x$ and $(10-x)$

$S(x) = x^2 + (10-x)^2 \Rightarrow$

$S(x) = x^2 + 100 - 20x + x^2$

$S(x) = 2x^2 - 20x + 100$

$S'(x) = 4x - 20 \Rightarrow S'(x) = 4(x-5)$

$4(x-5) = 0 \therefore x = 5$

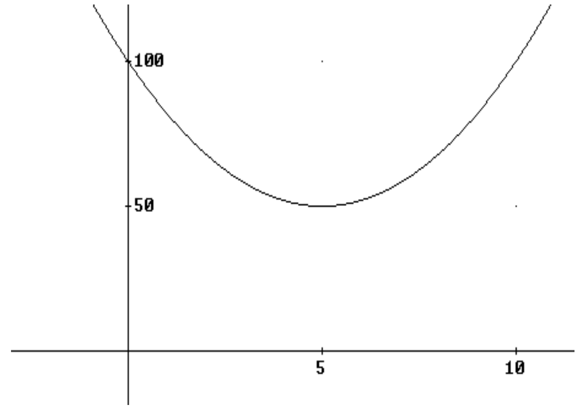
$S(0) = (0)^2 + (10-(0))^2 = 100$

$S(5) = (5)^2 + (10-(5))^2 = 50$

$S(10) = (10)^2 + (10-(10))^2 = 100$

Max sum is when numbers are 0 and 10

Min sum when numbers are 5 and 5



4. Find two positive numbers whose sum is 20 and such that its product is as large as possible. {10}

Numbers $= x$ and $(20 - x)$

$P(x) = x(20-x) \Rightarrow P(x) = 20x - x^2$

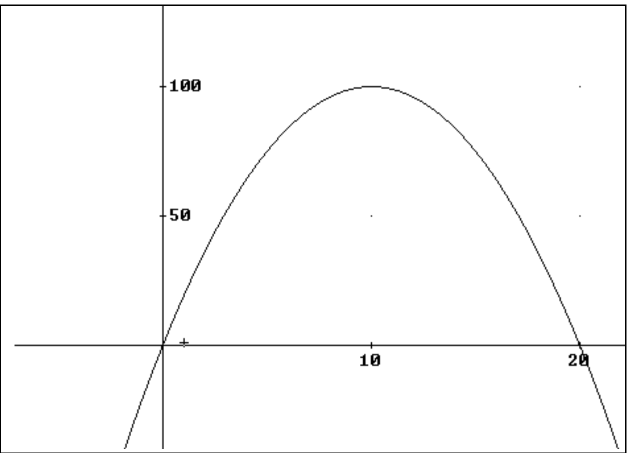
$P'(x) = 20 - 2x \Rightarrow P'(x) = 2(10-x)$

$2(10-x) = 0 \Rightarrow x = 10$

$P(0) = 0(20-0) = 0$

$P(10) = 10(20-10) = 100 \;\therefore$ max product

$P(20) = 20(20-20) = 0$



5. Find the rectangle with area 64 inches square for which the perimeter is a minimum. {8}

Length $= x$ and width $= y$

$A = xy = 64 \Rightarrow y = \dfrac{64}{x}$

$P = 2L + 2W \Rightarrow P = 2x + 2\left(\dfrac{64}{x}\right) \Rightarrow P = 2x + 128x^{-1}$

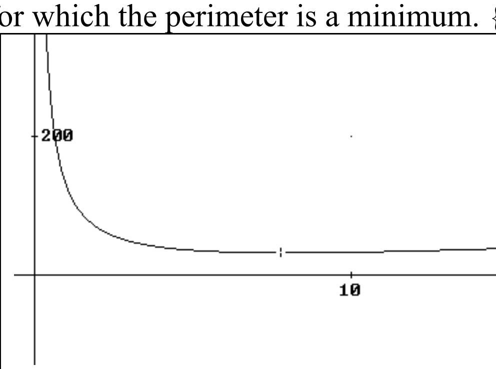
$P'(x) = 2 + 128 \cdot -1x^{-2} \Rightarrow P'(x) = 2 - 128x^{-2}$

$0 = 2 - 128x^{-2} \Rightarrow -1 = -128x^{-2} \Rightarrow x = 8$

Take the second derivative of P

$P''(x) = 256x^{-3} > 0$ when $x = 8$

P a minimum when $x = y = 8$



6. Find the point on the graph of $y = \sqrt{x}$ nearest the point (4, 0). $\left\{ \frac{7}{2}, \sqrt{\frac{7}{2}} \right\}$

Distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Points to consider: $(x_1, y_1) = (4,0)$ and $(x_2, y_2) = \left(x, \sqrt{x}\right)$

$d(x) = \sqrt{(x-4)^2 + (\sqrt{x} - 0)^2} \Rightarrow$

$d(x) = \sqrt{\left(x^2 - 8x + 16\right) + x} \Rightarrow$

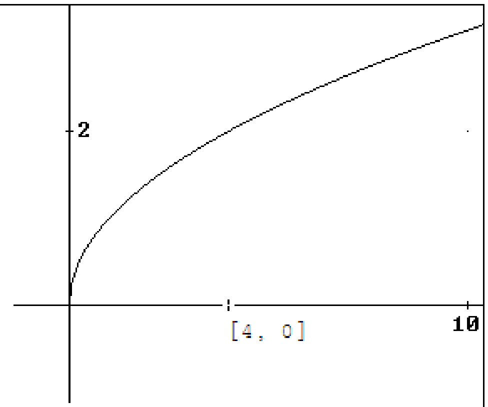
$d(x) = \sqrt{x^2 - 7x + 16} = \left(x^2 - 7x + 16\right)^{\frac{1}{2}}$

$d'(x) = \frac{1}{2}\left(x^2 - 7x + 16\right)^{\frac{1}{2}}(2x - 7)$

$0 = \frac{1}{2}\left(x^2 - 7x + 16\right)^{\frac{1}{2}}(2x - 7) \Rightarrow x = \frac{7}{2} \Rightarrow$ min x value

y value $d(x) = \sqrt{\left(\frac{7}{2}\right)^2 - 7\left(\frac{7}{2}\right) + 16} = \sqrt{\frac{7}{2}}$

Point closest $\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$



7. A rectangular field is to be fenced on three sides with 1000m of fencing (the fourth side being a straight river's edge). Find the dimensions of the field in order that the area be as large as possible. {250, 500}

Fencing = 1000 = $2x + y \Rightarrow y = 1000 - 2x$

$A = xy \Rightarrow A(x) = x\left(1000 - 2x\right) = 1000x - 2x^2$

$A'(x) = 1000 - 4x \Rightarrow 0 = 1000 - 4x \Rightarrow x = 250$

Domain [0, 500]

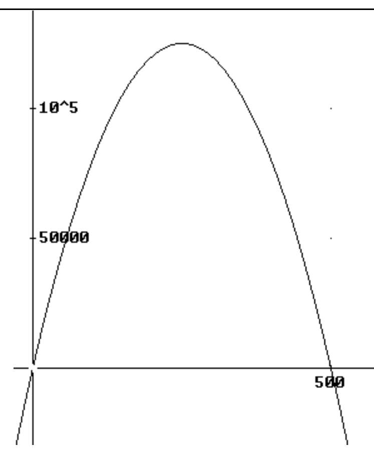
$A(0) = 1000(0) - 2(0)^2 = 0$

$A(0) = 1000(250) - 2(250)^2 = 125000$ $\Rightarrow$ Max area

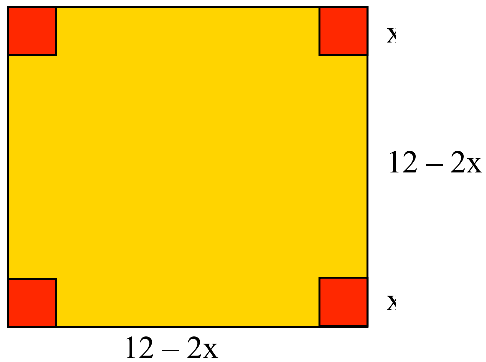
$A(0) = 1000(500) - 2(500)^2 = 0$

Dimensions: $\left(250, 500\right)$

$y = 1000 - 2x \Rightarrow y = 1000 - 2(25) = 500$



8. An open box is to be made from a square piece of cardboard measuring 12 inches on a side by cutting a square from each corner and folding up the sides. Find the dimensions for which the volume of the resulting box is a maximum. {2, 8, 8}



$V = lwh$

$V = (12-2x)(12-2x)x \Rightarrow V = 144x - 48x^2 + 4x^3$

$V' = 144 - 96x + 12x^2 \Rightarrow V' = 12(12 - 8x + x^2)$

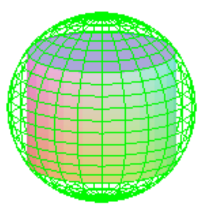
$0 = 12 - 8x + x^2 \Rightarrow 0 = (6-x)(2-x) \Rightarrow x = 2$ or 6

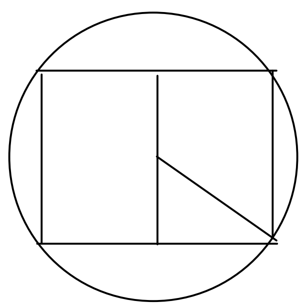
Eliminate $x = 6$ as a height since we would not have a box.

$V(2) = 144(2) - 48(2)^2 + 4(2)^3 \Rightarrow 128$

Dimensions = sides =$(12 - 2x)$=8

9. Find the right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. $\left\{\frac{10\sqrt{6}}{3}, \frac{20\sqrt{3}}{3}\right\}$





Volume of a right cylinder

$\Rightarrow V = \pi r^2 h$

Right Traingle Theorem

$10^2 = x^2 + \left(h/2\right)^2$

$100 = x^2 + h^2/4 \Rightarrow$

$\sqrt{400 - 4x^2} = h$

$V = \pi x^2 \sqrt{400 - 4x^2}$

$V' = 2\pi x \cdot \left(400 - 4x^2\right)^{\frac{1}{2}} + \frac{1}{2}\left(400 - 4x^2\right)^{-\frac{1}{2}}\left(-8x\right) \cdot \pi x^2$

$V' = 2\pi x\left(400 - 4x^2\right)^{-\frac{1}{2}}\left(400 - 4x^2 - 2x^2\right)$

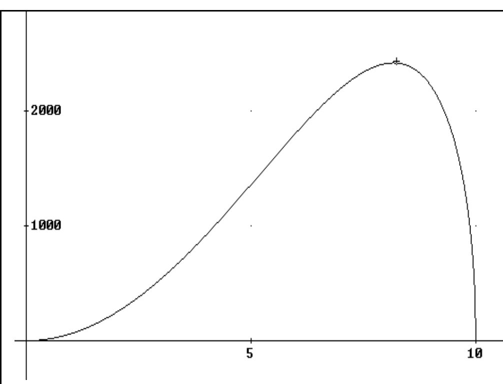
$V' = 2\pi x\left(400 - 4x^2\right)^{-\frac{1}{2}}\left(400 - 6x^2\right)$

$0 = 2\pi x\left(400 - 4x^2\right)^{-\frac{1}{2}}\left(400 - 6x^2\right)$

$0 = x$ and $0 = \left(400 - 6x^2\right)$

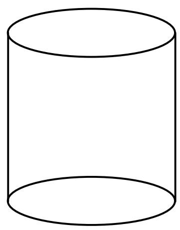
$x = 0$ and $x = \frac{10\sqrt{6}}{3}$

$h = \sqrt{400 - 4x^2} \Rightarrow h = \sqrt{400 - 4\left(\frac{10\sqrt{6}}{3}\right)^2} = \frac{20\sqrt{3}}{3}$



$V = \pi x^2 \sqrt{400 - 4x^2} = \frac{4000\pi\sqrt{3}}{9}$

10. Pop cans to hold 300ml are made in the shape of right-circular cylinders. Find the dimensions of the can that minimize its surface area. $\left\{\sqrt[3]{150/\pi}, 2\sqrt[3]{150/\pi}\right\}$



Two circles having a total area of

$A = \pi r^2(\text{top}) + \pi r^2(\text{bottom}) = 2\pi r^2$

Surface area is defined by the circumference of the can and the height $SA = 2\pi r \cdot h$

Total Surface Area $\Rightarrow SA = 2\pi r^2 + 2\pi r \cdot h$

Volume $\Rightarrow V = \pi r^2 \cdot h$

Volume $\Rightarrow V = \pi r^2 \cdot h$

$300 = \pi r^2 \cdot h \Rightarrow h = 300/\pi r^2$

$SA = 2\pi r^2 + 2\pi r \cdot 300/\pi r^2 \Rightarrow SA = 2\pi r^2 + 600r^{-1}$

$SA' = 4\pi r - 600r^{-2} \Rightarrow SA' = 4r^{-2}\left(\pi r^3 - 150\right)$

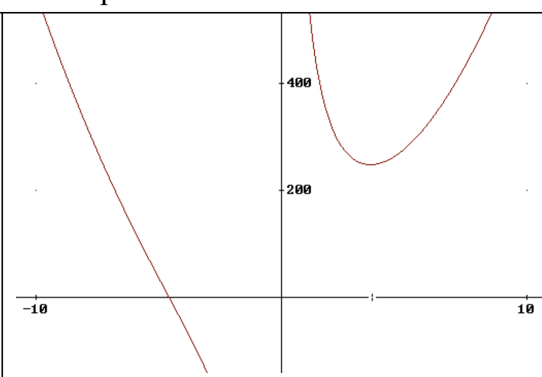
$\pi r^3 - 150 = 0 \Rightarrow \pi r^3 = 150 \Rightarrow r^3 = 150/\pi \Rightarrow \sqrt[3]{r^3} = \sqrt[3]{150/\pi} \Rightarrow r = \sqrt[3]{150/\pi}$

Second derivative because we do not have definite endpoints

$SA'' = 4\pi - 600\left(-2r^{-3}\right) \Rightarrow SA'' = 4\pi + 1200r^{-3} \Rightarrow$

$SA'' = 4\pi + 1200r^{-3} \Rightarrow SA'' = 4\pi + 1200/\sqrt[3]{150/\pi} > 0$

$\Rightarrow \therefore$ we have a local minimum



Determine the value of "h"

$$300 = \pi r^2 h \Rightarrow h = \frac{300}{\pi r^2} \Rightarrow h = \frac{300}{\pi\left(\sqrt[3]{\frac{150}{\pi}}\right)^2} = \frac{300 \cdot \sqrt[3]{\frac{150}{\pi}}}{\pi\left(\sqrt[3]{\frac{150}{\pi}}\right)^2 \cdot \sqrt[3]{\frac{150}{\pi}}} = \frac{300 \cdot \sqrt[3]{\frac{150}{\pi}}}{\pi \cdot \frac{150}{\pi}} = 2\sqrt[3]{\frac{150}{\pi}}$$

11. A length of wire L is to be cut into two pieces, one of which is bent to form a circle and the other to form a square. How should the wire be cut if the sum of the areas enclosed by the two pieces is a maximum? $\left\{\frac{L}{2\pi+8},\frac{L}{\pi+4}\right\}$

assume the side of the square is "x", the radius of the circle is "r"

Perimeter of the square = $4x$, Circumference of circle = $2\pi y$

Perimeter + Circumference = Total length of wire

Solving for the radius of the circle:

$$L = 4x + 2\pi y \Rightarrow x = \frac{L-2\pi y}{4} = \frac{L}{4} - \frac{\pi y}{2}$$

$$A = x^2 + \pi y^2 \Rightarrow A = \left(\frac{L}{4} - \frac{\pi y}{2}\right)^2 + \pi y^2 \Rightarrow$$

$$A' = 2\left(\frac{L}{4} - \frac{\pi y}{2}\right)\cdot\left(\frac{-\pi}{2}\right) + 2\pi y \Rightarrow A' = \frac{-\pi L}{4} + \frac{\pi^2 y}{2} + 2\pi y \Rightarrow$$

$$0 = \frac{-\pi L}{4} + \frac{\pi^2 y}{2} + 2\pi y \Rightarrow \frac{\pi L}{4} = \pi y\left(\frac{\pi}{2} + 2\right) \Rightarrow \frac{\pi L}{4} = \pi y\left(\frac{\pi+4}{2}\right) \Rightarrow$$

$$\frac{\pi L}{4}\cdot\frac{2}{(\pi+4)} = \pi y \Rightarrow \frac{\pi L}{2\pi}\cdot\frac{1}{(\pi+4)} = y \Rightarrow y = \frac{L}{(2\pi+8)}$$

Solving for the side of the square:

$$L = 4x + 2\pi y$$

$$L = 4x + 2\pi\left(\frac{L}{2\pi+8}\right) \Rightarrow L(2\pi+8) = 4x(2\pi+8) + 2\pi\left(\frac{L}{2\pi+8}\right)(2\pi+8)$$

$$2\pi L + 8L = 8\pi x + 32x + 2\pi L \Rightarrow 8L = x(8\pi+32) \Rightarrow x = \frac{8L}{(8\pi+32)} = \frac{L}{\pi+4}$$

Radius of circle exists between $\left[0, L/2\pi\right]$

If the entire "L" is used for the perimeter of the square then the side $= L/4$

$$\therefore A = x^2 \Rightarrow A = \left(L/4\right)^2 = L^2/16$$

From the first derivative $r = \left(L/(2\pi + 8) \right)$ and $x = L/(\pi + 4)$

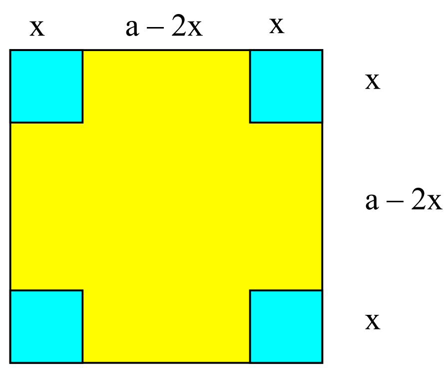
$$\therefore A = x^2 + \pi r^2 \Rightarrow A = \left(L/(\pi+4) \right)^2 + \pi \left(L/(2\pi+8) \right)^2 \Rightarrow A = \frac{L^2}{4(\pi+4)}$$

If the entire "L" is used for the circumference of the circle then $L = 2\pi r \Rightarrow r = \left(L/2\pi\right)$

$$\therefore A = \pi r^2 \Rightarrow A = \pi \left(L/2\pi\right)^2 \Rightarrow A = \frac{\pi L^2}{4\pi^2} = \frac{L^2}{4\pi} \Rightarrow \text{maximum area}$$

Recommend that you test the above by letting $L = 20$ (or some other value of your choosing)

12. A square sheet of tin "a" inches on a side is used to make an open-top box by cutting a small square of time from each corner and bending up the sides. How large a square should be cut from each corner in order that the box have as large a volume as possible? {a/6}



$V = lwh$

Consider "a" as some constant

$V = (a-2x)(a-2x)x \Rightarrow V = a^2x - 4ax^2 + 4x^3$

$V' = a^2 - 8ax + 12x^2 \Rightarrow V' = (a-6x)(a-2x)$

$0 = a - 6x \Rightarrow x = a/6$

$0 = a - 2x \Rightarrow x = a/2$

The domain would be on the interval [0, a/2]. The value of "x" cannot be larger than a/2 since that represents half the length of the sheet of square material.

$$V = a^2x - 4ax^2 + 4x^3$$

$$V = a^2(0) - 4a(0)^2 + 4(0)^3 = 0$$

$$V = a^2\left(a/6\right) - 4a\left(a/6\right)^2 + 4\left(a/6\right)^3 = a^3/6 - a^3/9 + a^3/54 = 4a^3/54 \Rightarrow \text{Maximum Volume}$$

$$V = a^2\left(a/2\right) - 4a\left(a/2\right)^2 + 4\left(a/2\right)^3 = a^3/2 - a^3 + a^3/2 = 0$$

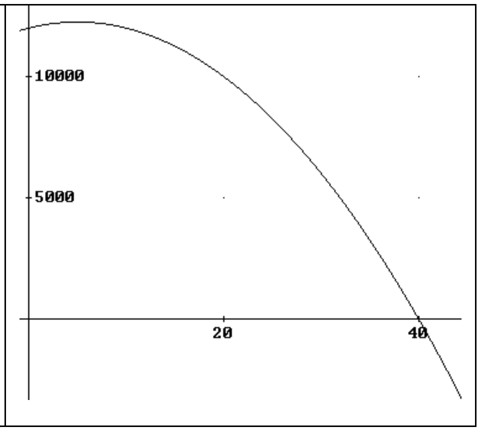
13. An apple orchard now has 30 trees per acre, and the average yield is 400 apples per tree. For each additional tree planted per acre, the average yield per tree is reduced by approximately 10 apples. How many trees per acre will give the largest crop of apples? {5 more}

Let x = the number of additional trees and y = the total yield of apples

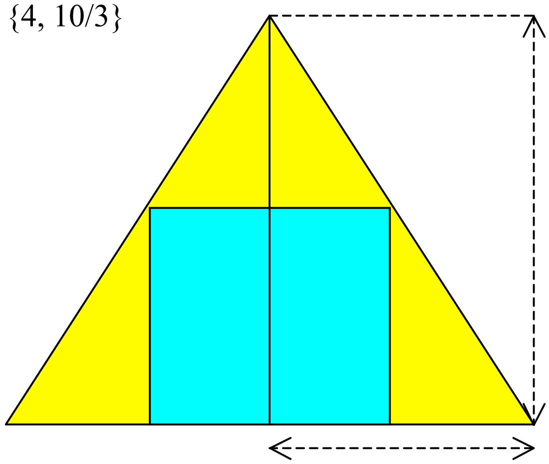
$$y = (30 + x)(400 - 10x)$$
$$y = 12000 - 300x + 400x - 10x^2$$
$$y = 12000 + 100x - 10x^2$$
$$y' = 100 - 20x$$
$$y' = 20(5 - x) \Rightarrow x = 5$$

because the function is continuous on the domain [0, 40] with "0" indicating no extra trees planted and "40" being the maximum number of trees that could be added which would result in average yield being reduced to zero.

$$y = \left(30 + 0\right)\left(400 - 10\left(0\right)\right) = 30 \cdot 400 = 12000$$
$$y = \left(30 + 5\right)\left(400 - 10\left(5\right)\right) = 35 \cdot 350 = 12250 \Rightarrow$$
maximum yield
$$y = \left(30 + 40\right)\left(400 - 10\left(40\right)\right) = 70 \cdot 0 = 0$$



14. Find the radius and height of the right-circular cylinder of largest volume that can be inscribed in a right circular cone with a radius of 6inches and a height of 10 inches. {4, 10/3}



Let r = radius of the cylinder and h = height of the cylinder

Apply similar triangles

$$\frac{h}{10} = \frac{6-r}{6} \Rightarrow h = \frac{60-10r}{6} \Rightarrow h = 10 - \frac{5r}{3}$$

The volume of the right cylinder: $V = \pi r^2 h$

To eliminate the variable h: $V = \pi r^2 \left(10 - \frac{5r}{3}\right) \Rightarrow V = 10\pi r^2 - \frac{5}{3}\pi r^3$

The domain of r: $\left[0,6\right]$ $\Rightarrow$ the radius of the cylinder can not be greater than the radius of the cone

$$V' = 10\pi \cdot 2r - \frac{5}{3}\pi \cdot 3r^2 \Rightarrow 20\pi r - 5\pi r^2$$

$$0 = 20\pi r - 5\pi r^2 \Rightarrow 0 = 5\pi r(4-r) \Rightarrow r = 0 \text{ or } r = 4$$

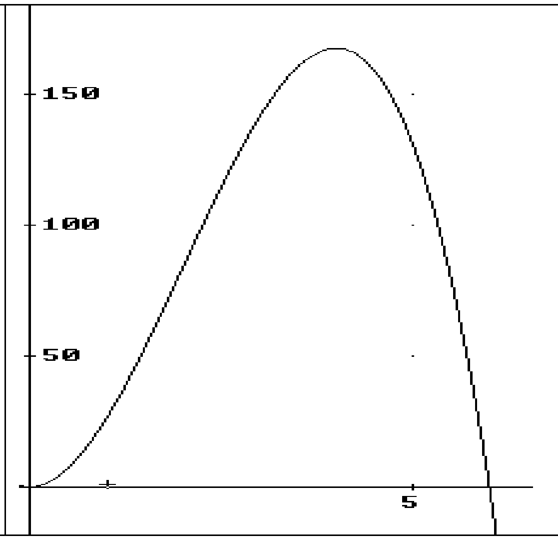
Because "V" is continuous on [0, 6] we can evaluate the function at its critical points and at the endpoints.

$$V = 10\pi r^2 - \frac{5}{3}\pi r^3$$

$$V(0) = 10\pi\left(0\right)^2 - \frac{5}{3}\pi\left(0\right)^3 = 0$$

$$V(4) = 10\pi\left(4\right)^2 - \frac{5}{3}\pi\left(4\right)^3 = \frac{160\pi}{3} \Rightarrow \text{maximum volume}$$

$$V(6) = 10\pi\left(6\right)^2 - \frac{5}{3}\pi\left(6\right)^3 = 0$$



The height of the right cylinder is:

$$h = 10 - \frac{5r}{3} \Rightarrow h = 10 - \frac{5(4)}{3} \Rightarrow h = \frac{30-20}{3} = \frac{10}{3}$$

15. A liquid from of penicillin manufactured by a pharmaceutical firm is sold in bulk at a price of \$200 per unit. If the production cost (in dollars) for "x" units is $c(x) = 500{,}000 + 80x + .003x^2$ and if the production capacity for the firm is at most 30,000 units in a specified time, how many units of penicillin must be manufactured and sold in that time to maximize profit? {20,000}

Profit(x) = Revenue(x) - Cost(x)

$$P(x) = 200x - \left(500{,}000 + 80x + .003x^2\right)$$

$$P(x) = 200x - 500{,}000 - 80x - .003x^2 \Rightarrow P(x) = -500{,}000 + 120x - .003x^2$$

$$P'(x) = 120 - .006x$$

$$120 - .006x = 0 \Rightarrow x = 20{,}000$$

Because "P(x)" is continuous on the interval [0, 30,000], we can evaluate the function at its critical points and at the endpoints

$$P(x) = -500{,}000 + 120x - .003x^2$$

$$P(x) = -500{,}000 + 120(0) - .003(0)^2 = -500{,}000$$

$$P(x) = -500{,}000 + 120(20000) - .003(20000)^2 = 700{,}000$$

$\Rightarrow$ maximum profit

$$P(x) = -500{,}000 + 120(30000) - .003(30000)^2 = 400{,}000$$